Homework V
Due Date: 14/04/2022

Exercise 1 (4 points). (1) Solve the 1D heat equation $\partial_t u = \partial_x^2 u$ in $0 < x < \ell$, with the mixed boundary conditions $u(t,0) = u(t,\ell) = 0$.
(2) Consider the equation $\partial_t^2 u = \partial_x^2 u$ for $0 < x < \ell$, with the boundary conditions $\partial_x u(t,0) = 0$, $u(t,\ell) = 0$ (Neumann at the left, Dirichlet at the right).
(a) Show that the eigenfunctions are $\cos\left(\frac{(n+\frac{1}{2})\pi x}{\ell}\right)$.
(b) Write the series expansion for a solution $u(t,x)$.
(3) Consider diffusion insider an enclosed circular tube. Let its length be 2ℓ. Let x denote the arc length parameter where $-\ell \le x \le \ell$. The concentration of the diffusing substance satisfies

$$\partial_t u = \partial_x^2 u \quad \text{for } -\ell \le x \le \ell,$$
$$u(t,-\ell) = u(t,\ell) \quad \text{and} \quad \partial_x u(t,-\ell) = \partial_x u(t,\ell).$$

These are called periodic boundary conditions.
(a) Show that the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$ for $n = 0, 1, 2, \dots$
(b) Show that the solution is

$$u(t,x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{\ell} + B_n \sin\frac{n\pi x}{\ell}\right) e^{-\frac{n^2\pi^2 t}{\ell^2}}.$$

Exercise 2 (4 points). Consider the unusual eigenvalue problem

$$-\partial_x^2 v = \lambda v, \quad \text{for } 0 < x < \ell,$$
$$\partial_x v(0) = \partial_x v(\ell) = \frac{v(\ell) - v(0)}{\ell}.$$

(a) Show that $\lambda = 0$ is a double eigenvalue.
(b) Get an equation for the positive eigenvalues $\lambda > 0$.
(c) Letting $\gamma = \frac{1}{2}\ell\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.
(e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenvalues and eigenfunctions.
(f) Solve the problem $\partial_t u = \partial_x^2 u$ for $0 < x < \ell$, with the BCs given above, and with $u(0,x) = \phi(x)$.
(g) Show that $t \to \infty$, $\lim u(t,x) = A + Bx$ for some constants A and B, assuming that you can take limits term by term.

Exercise 5 (2 points) (a). Find the positive eigenvalues and the corresponding eigenfunctions of the fourth-order operator $\frac{\mathrm{d}^4}{\mathrm{d}x^4}$ with the four boundary conditions

$$X(0) = X(\ell) = X''(0) = X''(\ell) = 0.$$

(b). Solve the fourth-order eigenvalue problem $X'''' = \lambda X$ in $0 < x < \ell$, with the four boundary conditions,

$$X(0) = X'(0) = X(\ell) = X'(\ell) = 0,$$

where $\lambda > 0$.